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XXI.

INVESTIGATIONS IN QUATERNIONS.

THE THESIS OF A CANDIDATE FOR MATHEMATICAL HONORS CONFERRED WITH THE DEGREE OF A.B., AT HARVARD COLLEGE, AT COMMENCEMENT, 1877.

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Presented by Professor Benjamin Peirce, Jan. 9, 1878.

I. Logarithms of Quaternions.

1°. Hamilton proves (Lectures, p. 546) the following theorem: The tensor of the sum of any number of quaternions cannot exceed the sum of their tensors.

If p and q be any two quaternions, then

$$\begin{aligned} \mathrm{T}(p+q)^2 &= (p+q)(\mathrm{K}p+\mathrm{K}q) = \mathrm{T}p^2+\mathrm{T}q^2+2\mathrm{S}.p\mathrm{K}q \\ &= \mathrm{T}p^2+\mathrm{T}q^2+2\mathrm{T}p\mathrm{T}q\mathrm{SU}.p\mathrm{K}q \\ &= (\mathrm{T}p+\mathrm{T}q)^2-2\mathrm{T}p\mathrm{T}q(1-\mathrm{SU}.p\mathrm{K}q)\,; \end{aligned}$$

and since the scalar of a versor cannot fall outside the limits ± 1,

$$\therefore\ \mathrm{T}(p+q) \mathrel{-\!\!<} \mathrm{T}p+\mathrm{T}q.$$

The same proof holds for any number of quaternions, and gives in general

$$\mathrm{T}(p+q+r+\ldots) \mathrel{-\!\!<} \mathrm{T}p+\mathrm{T}q+\mathrm{T}r+\ldots$$

On the foregoing theorem depends the test of convergence of the series

$$\Sigma = 1+q+\frac{q^2}{2\,!}+\frac{q^3}{3\,!}+\cdots$$

Let Σ_n represent the first n terms of Σ; $S =$ the series formed by the tensors of Σ; $S_n =$ the first n terms of S. The terms of S are represented by the general form $\frac{\mathrm{T}q^n}{n\,!} = a_n$. Each successive term of this series is formed by multiplying the preceding term by a factor of the form $\frac{Tq}{n}$. But n continually increases, while Tq remains constant. Hence, however great a finite value Tq may have, n will finally become greater than Tq, and still remain finite. It will therefore be possible to take n so great, and still finite, that the ratio between the last two terms of S_n shall be finitely greater than unity. Insert k, a finite quantity, between this limit and unity. Then

$$\frac{a_n}{a_{n+1}} > k > 1, \qquad \text{whence}$$

$a_{n+1} < \frac{a_n}{k}$, $a_{n+2} < \frac{a_{n+1}}{k} < \frac{a_n}{k^2}$, and in general $a_{n+p} < \frac{a_n}{k^p}$. Then, designating by $S_{n,p}$ the p terms from a_n to a_{n+p-1} inclusive, we have

$$S_{n,p} = a_n + a_{n+1} + \dots + a_{n+p-1} < a_n \left(1 + \frac{1}{k} + \dots + \frac{1}{k^{p-1}}\right)$$

or

$$S_{n,p} < a_n \left(\frac{1}{1-\frac{1}{k}} - \frac{\frac{1}{k^p}}{1-\frac{1}{k}}\right), \quad S_{n,p} < a_n \left(\frac{k}{k-1} - \frac{k}{k^p(k-1)}\right).$$

When $p = \infty$, $S_{n,p} < a_n \frac{k}{k-1}$, and therefore $S_{n,\infty}$ is finite, and since S_n is finite, therefore $S_n + S_{n,\infty} = S$ is finite, and S is convergent. But $T\Sigma \mathrel{-\!\!<} S$ and $\therefore$ $T\Sigma$ is finite. If the tensor of a quaternion is finite, the quaternion is finite; and hence, as was to be proved, the series

$$\Sigma = 1 + q + \frac{q^2}{2!} + \dots$$

is convergent. This series is the exponential of quaternions, and may be designated by the usual symbol ϭ^{q}. It is a quaternion complanar with q.

2°. In ordinary scalar analysis, the fundamental principle of the exponential function consists in the relation

$$\text{ϭ}^{q+q'} = \text{ϭ}^{q}\text{ϭ}^{q'},$$

where q and q' are numbers. In quaternion analysis, this principle holds good only for complanar quaternions, in which the commutative property always obtains. The proof I give is precisely similar to that given for complex scalar quantities by MM. Briot et Bouquet, Fonc. Ellipt. p. 89.

Let Σ_n, Σ'_n, Σ''_n, represent the sums of the first n terms of the series which define the functions $\mathcal{E}^q$, $\mathcal{E}^{q'}$, $\mathcal{E}^{q+q'}$: —

$$\Sigma_n = 1 + q + \frac{q^2}{2\,!} + \ldots \frac{q^{n-1}}{(n-1)\,!},$$

$$\Sigma'_n = 1 + q' + \frac{q'^2}{2\,!} + \ldots \frac{q'^{n-1}}{(n-1)\,!},$$

$$\Sigma''_n = 1 + (q + q') + \ldots \frac{(q + q')^{n-1}}{(n-1)\,!}.$$

Also, represent the functions $\mathcal{E}^{\mathrm{T}q}$, $\mathcal{E}^{\mathrm{T}q'}$, $\mathcal{E}^{\mathrm{T}q+\mathrm{T}q'}$, down to n terms, by

$$S_n = 1 + \mathrm{T}q + \frac{\mathrm{T}q^2}{2\,!} + \ldots \frac{\mathrm{T}q^{n-1}}{(n-1)\,!},$$

$$S'_n = 1 + \mathrm{T}q' + \frac{\mathrm{T}q'^2}{2\,!} + \ldots \frac{\mathrm{T}q'^{n-1}}{(n-1)\,!},$$

$$S''_n = 1 + (\mathrm{T}q + \mathrm{T}q') + \ldots \frac{(\mathrm{T}q + \mathrm{T}q')^{n-1}}{(n-1)\,!}.$$

Of this last series, take the sum of the first $(2n-1)$ terms: —

$$S'_{2n-1} = 1 + (\mathrm{T}q + \mathrm{T}q') + \ldots + \frac{(\mathrm{T}q + \mathrm{T}q')^{2n-2}}{(2n-2)\,!}.$$

By developing the numerators of S''_n and S''_{2n-1}, and effecting the product $S_nS'_n$, we shall discover (since the multiplications are commutative) that all the terms of S''_n are to be found in the product $S_nS'_n$, and that all the terms of this product are completely represented by corresponding terms in S''_{2n-1}. Hence

$$S'_n < S_n'S_n < S''_{2n-1}.$$

When n increases indefinitely, S''_n and S''_{2n-1} tend to the same limit $\mathcal{E}^{\mathrm{T}q+\mathrm{T}q'}$;

$$\therefore \lim_{n=\infty} (S_nS'_n - S''_n) = 0.$$

The property, that the tensor of the sum of any number of quaternions is as small as the sum of the tensors, together with the fact that the terms of the expression $(\Sigma_n\Sigma'_n - \Sigma''_n)$ have for their tensors the corresponding terms of $(S_nS'_n - S''_n)$, gives

$$\mathrm{T}(\Sigma_n\Sigma'_n - \Sigma''_n) \mathrel{-\!\!<} S_nS'_n - S''_n,$$

and

$$\therefore \lim_{n=\infty} \mathrm{T}(\Sigma_n\Sigma'_n - \Sigma''_n) = 0.$$

Now when Tq is null, q itself is null, and

$$\therefore \lim\,(\Sigma_n\Sigma'_n - \Sigma''_n) = 0.$$

Thus at the limit $\Sigma_n\Sigma'_n = \Sigma''_n$,

or $\epsilon^q\epsilon^{q'} = \epsilon^{q+q'}$. (1)

This proof includes the proof of $\epsilon^q = \epsilon^{\mathrm{S}q}\epsilon^{\mathrm{V}q}$, since in the products of scalars with vectors the commutative property holds.

With diplanar quaternions, formula (1) is not true, since such quaternions are not commutative with each other.

3°. Defining the five quaternion functions ϵ^q, sin q, cos q, Sh q, Ch q, by the five series

$$(a)\quad \epsilon^q = \sum_0^\infty \frac{q^n}{n\,!} = 1 + q + \frac{q^2}{2\,!} + \frac{q^3}{3\,!} + \cdots$$

$$(b)\quad \sin q = \sum_0^\infty (-)^n \frac{q^{2n+1}}{(2n+1)\,!} = q - \frac{q^3}{3\,!} + \frac{q^5}{5\,!} - \cdots$$

$$(c)\quad \cos q = \sum_0^\infty (-)^n \frac{q^{2n}}{(2n)\,!} = 1 - \frac{q^2}{2\,!} + \frac{q^4}{4\,!} - \cdots$$

$$(d)\quad \mathrm{Sh}\ q = \sum_0^\infty \frac{q^{2n+1}}{(2n+1)\,!} = q + \frac{q^3}{3\,!} + \frac{q^5}{5\,!} + \cdots$$

$$(e)\quad \mathrm{Ch}\ q = \sum_0^\infty \frac{q^{2n}}{(2n)\,!} = 1 + \frac{q^2}{2\,!} + \frac{q^4}{4\,!} + \cdots$$

we have, by adding (d) and (e)

$$\epsilon^q = \mathrm{Ch}\ q + \mathrm{Sh}\ q, \qquad (2)$$

and by taking for q the particular value Vq,

$$\epsilon^{\mathrm{V}q} = \mathrm{ChV}q + \mathrm{ShV}q\,;$$

whence observing that $\epsilon^q = \epsilon^{\mathrm{S}q}\epsilon^{\mathrm{V}q}$, we obtain the important general formula

$$\epsilon^q = \epsilon^{\mathrm{S}q}\,(\mathrm{ChV}q + \mathrm{ShV}q). \qquad (3)$$

If i denotes any value of $\sqrt{-1}$ which is commutative with q, formulae (b), (c), (d), and (e) give

$$\sin qi = i \,\mathrm{Sh}\, q, \qquad \cos qi = \mathrm{Ch}\, q,$$

$$\mathrm{Sh}\, qi = i \sin q, \qquad \mathrm{Ch}\, qi = \cos q;$$

and (2) gives

$$\epsilon^{qi} = \cos q + i \sin q. \tag{4}$$

Thus, taking $\mathrm{TV}q$ for q, and $\mathrm{UV}q$ for i, we can write formula (3) as follows: —

$$\epsilon^{q} = \epsilon^{\mathrm{S}q} (\cos \mathrm{TV}q + \mathrm{UV}q \,.\, \sin \mathrm{TV}q). \tag{5}$$

In the special case of a vector, (3) and (5) become

$$\epsilon^{\alpha} = \mathrm{Ch}\, \alpha + \mathrm{Sh}\, \alpha \tag{3$'$}$$

$$\epsilon^{\alpha} = \cos \mathrm{T}\alpha + \mathrm{U}\alpha \sin \mathrm{T}\alpha, \tag{5$'$}$$

where α is any vector whatever.

4°. Formulæ, analogous to those just given, may be deduced for biquaternions.

Let $\mathrm{Q} = q_1 + iq_2$, where $i = \sqrt{-1}$ (a scalar), and q_1 and q_2 are complanar. We have $\mathrm{UV}q_2 = \pm\, \mathrm{UV}q_1$. Since i is commutative with any quaternion, we have, by (4), —

$$\epsilon^{i\mathrm{V}q_2} = (\cos \mathrm{V}q_2 + i \sin \mathrm{V}q_2),$$

which, multiplied by (5) and by $\epsilon^{i\mathrm{S}q_2}$, gives

$$\epsilon^{q_1 + iq_2} = \epsilon^{\mathrm{S}(q_1 + iq_2)} (\cos \mathrm{TV}q_1 + \mathrm{UV}q_1 \sin \mathrm{TV}q_1)(\cos \mathrm{V}q_2 + i \sin \mathrm{V}q_2)$$

$$= \epsilon^{\mathrm{SQ}} (\cos \mathrm{TV}q_1 + \mathrm{UV}q_1 \sin \mathrm{TV}q_1)(\mathrm{Ch}\, \mathrm{TV}q_2 + i\mathrm{UV}q_2 \,\mathrm{Sh}\, \mathrm{TV}q_2)$$

$$= \epsilon^{\mathrm{SQ}} \begin{bmatrix} \cos \mathrm{TV}q_1 \,\mathrm{Ch}\, \mathrm{TV}q_2 \pm i\mathrm{UV}q_1 \cos \mathrm{TV}q_1 \,\mathrm{Sh}\, \mathrm{TV}q_2 \\ \mp\, i \sin \mathrm{TV}q_1 \,\mathrm{Sh}\, \mathrm{TV}q_2 + \mathrm{UV}q_1 \sin \mathrm{TV}q_1 \,\mathrm{Ch}\, \mathrm{TV}q_2 \end{bmatrix};$$

$$\epsilon^{\mathrm{Q}} = \epsilon^{\mathrm{SQ}} [\cos (\mathrm{TV}q_1 \pm i\mathrm{TV}q_2) + \mathrm{UV}q_1 \sin (\mathrm{TV}q_1 \pm i\mathrm{TV}q_2)]. \tag{6}$$

Observing that

$$\cos (\mathrm{TV}q_1 \pm i\mathrm{TV}q_2) = \mathrm{Ch}\, \mathrm{V}(q_1 + iq_2),$$

and that

$$\sin (\mathrm{TV}q_1 \pm i\mathrm{TV}q_2) = -\, \mathrm{UV}q_1 \,\mathrm{Sh}\, \mathrm{V}(q_1 + iq_2),$$

and substituting these values in (6), we obtain this other important formula

$$\epsilon^{\mathrm{Q}} = \epsilon^{\mathrm{SQ}} (\mathrm{Ch}\, \mathrm{VQ} + \mathrm{Sh}\, \mathrm{VQ}). \tag{7}$$

For bivectors, we have, if $\alpha = \alpha_1 + \mathrm{i}\alpha_2$, provided that α_1 and α_2 are parallel,

$$\text{ϭ}^{\alpha} = \cos\,(\mathrm{T}\alpha_1 + \mathrm{i}\mathrm{T}\alpha_2) + \mathrm{U}\alpha \sin\,(\mathrm{T}\alpha_1 + \mathrm{i}\mathrm{T}\alpha_2),$$

$$= \mathrm{Ch}\,\alpha + \mathrm{Sh}\,\alpha.$$

5°. The usual algebraic definition of the natural logarithm, applied to a quaternion, gives

$$\text{ϭ}^{\log q} = q. \qquad (8)$$

Let $$\log q = p, \log q' = p'.$$

$$\therefore\ \text{ϭ}^{p} = q,\ \text{ϭ}^{p'} = q',$$

and by (1) $$\text{ϭ}^{p+p'} = qq',$$

which gives, for complanar (but not for diplanar) quaternions, the general formula

$$\log qq' = \log q + \log q'. \qquad (9)$$

This formula, however, is subject to certain limitations as will be shown later, in 9°. But in particular and always, we have

$$\log q = \log \mathrm{T}q\mathrm{U}q = \log \mathrm{T}q + \log \mathrm{U}q. \qquad (10)$$

6°. Let $\log\mathrm{U}q = q$, where q' is an undetermined quaternion. By (5)

$$\text{ϭ}^{q'} = \mathrm{U}q = \text{ϭ}^{\mathrm{S}q'}(\cos \mathrm{TV}q' + \mathrm{UV}q' \sin \mathrm{TV}q'),$$

whence,

$$\mathrm{SU}q = \text{ϭ}^{\mathrm{S}q'} \cos \mathrm{TV}q' = \cos \angle q,$$

$$\mathrm{TVU}q = \mathrm{T}.\ \text{ϭ}^{\mathrm{S}q'} \sin \mathrm{TV}q' = \sin \angle q\,;$$

$$\therefore\ \mathrm{T}^2\mathrm{VU}q + \mathrm{S}^2\mathrm{U}q' = \text{ϭ}^{2\mathrm{S}q'} = 1,$$

$$\therefore\ \text{ϭ}^{\mathrm{S}q'} = (-)^n\, 1, \text{ and } \mathrm{S}q' = n\, \text{ə}\, \mathrm{i};$$

where i is the scalar $\sqrt{-1}$, and n any integer. Comparing these results, we find

$$\cos \angle q = (-)^n \cos \mathrm{TV}q',\ \sin \angle q = \mathrm{T} \sin \mathrm{TV}q',$$

$$\pm\, \mathrm{TV}q' = \angle q + k\, \text{ə}, \qquad \mathrm{V}q' = \pm\, (\angle q + k\, \text{ə})\mathrm{UV}q',$$

where k is any integer which differs from n by an even number. But

$$\pm\, \mathrm{UV}q' = \mathrm{UVU}q, \text{ and } \mathrm{V}q \parallel \mathrm{VU}q,$$

and therefore

$$\mathrm{V}q' = (\angle q + k\,\partial)\mathrm{UV}q,$$

and since

$$\mathrm{S}q' = n\,\partial\,\mathrm{i},$$

$$\therefore\ q' = n\,\partial\,\mathrm{i} + (\angle q + k\,\partial)\mathrm{UV}q,$$

whence follows the fundamental formula

$$\log q = \log \mathrm{T}q + n\,\partial\,\mathrm{i} + (\angle q + k\,\partial)\mathrm{UV}q; \qquad (11)$$

which can be more accurately written

$$\log q = \log \mathrm{T}q + n\,\partial\,\mathrm{i} + [\angle q + (n + 2k)\,\partial]\mathrm{UV}q. \qquad (12)$$

If q be negative, this becomes

$$\log(-q) = \log \mathrm{T}q + n\partial\mathrm{i} + [\angle q + (n + 2k + 1)\partial]\mathrm{UV}q. \quad (13)$$

If q be a vector, its angle is $\frac{\partial}{2}$ and

$$\log(\pm\,\alpha) = \log \mathrm{T}\alpha + n\,\partial\,\mathrm{i} \pm \frac{2(n + 2k) + 1}{2}\,\partial\mathrm{U}\alpha. \quad (14)$$

If $n = k = 0$, and if the *real* logarithm of $\mathrm{T}q$ be taken, we have what Hamilton calls the *principal* logarithm, or simply *the Logarithm*, of a quaternion, and indicates by using l instead of log. Thus we have

$$\mathrm{l}q = \mathrm{lT}q + \angle q\mathrm{UV}q. \qquad (12)'$$

The principal logarithm of q is a real quaternion complanar with q itself. Its scalar and vector parts are

$$\mathrm{Sl}q = \mathrm{lT}q,\ \mathrm{Vl}q = \angle q\mathrm{UV}q.$$

7°. The formulae of 4° render it possible to find the logarithm of a biquaternion. Let

$$\log \mathrm{UQ} = \mathrm{Q}' = q'_1 + \mathrm{i}q'_2,$$

where Q' is an undetermined biquaternion, and $\mathrm{Q} = q_1 + \mathrm{i}q_2$ is such that $\mathrm{UV}q_1 = \mathrm{UV}q_2$. Formula (6) gives

$$\mathrm{SUQ} = \text{ϭ}^{\mathrm{SQ}'} \cos(\mathrm{TV}q'_1 + \mathrm{iTV}q'_2),$$

$$\mathrm{TVUQ} = \mathrm{T}\text{ϭ}^{\mathrm{SQ}'} \sin(\mathrm{TV}q'_1 + \mathrm{iTV}q'_2),$$

$$\therefore\ \mathrm{T}^2\mathrm{VUQ} + \mathrm{S}^2\mathrm{UQ} = \text{ϭ}^{2\mathrm{SQ}'} = 1. \qquad (15)$$

Let us agree to define the angle of Q as such that

$$\mathrm{SUQ} = \cos \angle \mathrm{Q}, \text{ and } \mathrm{TVUQ} = \sin \angle \mathrm{Q}.$$

This definition will be admissible if it be not found inconsistent with other established definitions. Then (15) gives

$$\mathrm{SQ}' = n\, \text{ə}\, \mathrm{i},$$

and

$$\angle \mathrm{Q} = \mathrm{TV}q'_1 + \mathrm{iTV}q'_2 + k\, \text{ə},$$

n and k being any integers, which differ by an even number. Multiplying $\angle \mathrm{Q}$ into $\mathrm{UV}q_1$, we have

$$\angle \mathrm{Q}.\mathrm{UV}q_1 = \mathrm{V}(q'_1 + \mathrm{i}q'_2) + k\text{ə}\mathrm{UV}q_1,$$

$$(\angle \mathrm{Q} + k\text{ə})\mathrm{UV}q_1 = \mathrm{V}(q'_1 + \mathrm{i}q'_2) = \mathrm{VQ}',$$

or $$\mathrm{Q}' = n\text{ə}\mathrm{i} + [\angle \mathrm{Q} + (n + 2\,k)\text{ə}]\mathrm{UV}q_1 = \log \mathrm{UQ}.$$

Whence, for the general logarithm of a biquaternion, of which the real components are complanar, we find

$$\begin{aligned}\log \mathrm{Q} &= \log \mathrm{TQ} + \log \mathrm{UQ}\\ &= \log \mathrm{TQ} + n\text{ə}\mathrm{i} + [\angle \mathrm{Q} + (n+2k)\text{ə}]\mathrm{UV}q_1; \quad (16)\end{aligned}$$

and, for the principal logarithm of such a biquaternion,

$$\mathrm{l\,Q} = \mathrm{l{\cdot}TQ} + (\angle \mathrm{Q})\mathrm{UV}q_1. \qquad (17)$$

8°. For the product of any number of complanar quaternions (when the sum of their angles does not exceed 180°) we have by (9)

$$\log (q.\, q'.\, q''. \ldots) = \log q + \log q' + \ldots,$$

or

$$\log \Pi q = \Sigma \log q; \qquad (18)$$

and since $\mathrm{UV}q = \mathrm{UV}q' = \mathrm{UV}q''$ etc., we therefore obtain at once from (12) the general formula

$$\begin{aligned}\log \Pi q &= \Sigma \log \mathrm{T}q + n\text{ə}\mathrm{i} + [\Sigma\angle q + (n + 2k)\text{ə}]\mathrm{UV}q\\ &= \log \Pi\mathrm{T}q + n\text{ə}\mathrm{i} + [\angle \Pi q + (n + 2k)\text{ə}]\mathrm{UV}q. \quad (19)\end{aligned}$$

This formula gives the principal logarithm only when $(\Sigma\angle q) < 180°$ (9°). The two expressions of this formula show that in general ($\Sigma\angle q$ being $<180°$),

$$\angle \Pi q = \Sigma \angle q;$$

which is otherwise easily seen to be true.

9°. The angle of a quaternion has not the usual generality of an angle in trigonometry; it is never negative or greater than 180°. (19) must, then, be modified as follows, to give the principal logarithm. It is easily seen that if

$$(\Sigma\angle q) > 2m\text{ə} \text{ and } < (2m+1)\text{ə},$$

$$\angle \Pi q = \Sigma\angle q - 2m\text{ə},$$

and that, if

$$(\Sigma\angle q) > (2m+1)\text{ə} \text{ and } < (2m+2)\text{ə},$$

$$\angle \Pi q = (2m+2)\text{ə} - \Sigma\angle q.$$

Hence, the principal logarithm of Πq becomes

$$\mathrm{l}\Pi q = \Sigma\,\mathrm{lT}q + (\Sigma\angle q - 2m\text{ə})\mathrm{UV}q, \qquad (20)$$

$$\text{if } (\Sigma\angle q) > 2m\text{ə} \text{ and } < (2m+1)\text{ə};$$

or

$$\mathrm{l}\Pi q = \Sigma\,\mathrm{lT}q + [\Sigma\angle q - (2m+2)\text{ə}]\mathrm{UV}q, \qquad (20)'$$

$$\text{if } (\Sigma\angle q) > (2m+1)\text{ə} \text{ and } < (2m+2)\text{ə}.$$

For the power of a quaternion, these formulae become

$$\mathrm{l}q^n = n\mathrm{lT}q + (n\angle q - 2m\text{ə})\mathrm{UV}q, \qquad (21)$$

$$\text{if } (n\angle q) > 2m\text{ə} \text{ and } < (2m+1)\text{ə};$$

or

$$\mathrm{l}q^n = n\mathrm{lT}q + [n\angle q - (2m+2)\text{ə}]\mathrm{UV}q, \qquad (21)'$$

$$\text{if } (n\angle q) > (2m+1)\text{ə} \text{ and } < (2m+2)\text{ə}.$$

For the principal logarithm of any integral power of a vector: —

$$\mathrm{l}\alpha^{4n} = 4n\mathrm{lT}\alpha,$$

$$\mathrm{l}\alpha^{4n+1} = (4n+1)\mathrm{lT}\alpha + \tfrac{1}{2}\text{ə}\mathrm{U}\alpha,$$

$$\mathrm{l}\alpha^{4n+2} = (4n+2)\mathrm{lT}\alpha \pm \text{ə}\mathrm{U}\alpha,^*$$

$$\mathrm{l}\alpha^{4n+3} = (4n+3)\mathrm{lT}\alpha - \tfrac{1}{2}\text{ə}\mathrm{U}\alpha.$$

The cases in which $\mathrm{l}q^n = n\mathrm{l}q$ are those in which $(n\angle q) < \text{ə}$.

For $\mathrm{l}q^{\frac{1}{n}}$ we shall always have

$$\mathrm{l}q^{\frac{1}{n}} = \tfrac{1}{n}\mathrm{lT}q + \tfrac{1}{n}\angle q.\mathrm{UV}q,$$

---

* But for $\mathrm{U}\alpha$ in this formula, *any* unit-vector may be substituted.

since $(\angle q) < \partial$ and $\therefore$ $(\frac{1}{n}\angle q) < \partial$; hence, in general,

$$\mathrm{l}q^{\frac{1}{n}} = \frac{1}{n}\mathrm{l}q.$$

Hence $$\mathrm{l}q^{\frac{m}{n}} = \frac{m}{n}\mathrm{l}q, \text{ if } (\tfrac{m}{n}\angle q) < \partial.$$

Again $\angle q^{-1} = \angle q$, $\mathrm{UV}q^{-1} = -\mathrm{UV}q$; and hence

$$\mathrm{l}q^{-1} = -\mathrm{lT}q - \angle q.\mathrm{UV}q,$$

$$\mathrm{l}q^{-n} = -n\mathrm{lT}q - n\angle q.\mathrm{UV}q, \text{ if } (n\angle q) < \partial.$$

We see then that, in general, n being integral or fractional, positive or negative, and q being any quaternion,

$$\mathrm{l}q^{n} = n\mathrm{l}q, \text{ if } -\partial < (n\angle q) < +\partial. \tag{22}$$

10°. The following results are deduced immediately from those already obtained:—

$$\begin{aligned}\mathrm{l}(-q) &= \mathrm{lT}q - (\partial - \angle q)\mathrm{UV}q \\ &= \mathrm{l}q - \partial\mathrm{UV}q = \mathrm{l}q + \partial\mathrm{UV}(-q),\end{aligned} \tag{23}$$

$$\mathrm{lK}q = \mathrm{lT}q - \angle q.\mathrm{UV}q = \mathrm{Kl}q, \tag{24}$$

$$\mathrm{l}q^{-1} = -\mathrm{lT}q - \angle q.\mathrm{UV}q = -\mathrm{l}q, \tag{25}$$

$$\mathrm{l}\frac{\alpha}{\beta} = \mathrm{lT}\alpha - \mathrm{lT}\beta + \angle\frac{\alpha}{\beta}\mathrm{UV}\beta\alpha, \tag{26}$$

$$\mathrm{l}\alpha\beta = \mathrm{lT}\alpha + \mathrm{lT}\beta + (\partial - \angle\frac{\beta}{\alpha})\mathrm{UV}\alpha\beta. \tag{27}$$

11°. Suppose a series of diplanar quaternions to be such that

$$\mathrm{U}q = \mathrm{U}\frac{\beta}{\alpha},\ \mathrm{U}q' = \mathrm{U}\frac{\gamma}{\beta},\ \mathrm{U}q'' = \mathrm{U}\frac{\delta}{\gamma}, \text{ etc.},$$

where $\alpha, \beta, \gamma, \delta, \ldots \omega$, are any vectors whatever. Then

$$\mathrm{U}\delta = \mathrm{U}q''\gamma,\ \mathrm{U}\delta\beta^{-1} = \mathrm{U}q''\gamma\beta^{-1} = \mathrm{U}q''q',$$

$$\mathrm{U}q''q'q = \mathrm{U}\delta\beta^{-1}\,\beta\alpha^{-1} = \mathrm{U}\delta\alpha^{-1},$$

and, continuing this process, we shall finally have

$$\mathrm{U}(q_\omega \ldots q''q'q) = \mathrm{U}\frac{\omega}{\alpha},$$

where ω is the last vector of the series. Hence, the product of such a series of quaternions will be

$$q_\omega \ldots q''q'q = \mathrm{T}\,(q_\omega \ldots q''q'q)\mathrm{U}\frac{\omega}{\alpha}.$$

The principal logarithm of such a product is

$$\mathrm{l}(q_\omega \ldots q''q'q) = \Sigma\mathrm{lT}q + \angle_{\alpha}^{\omega}\,\mathrm{UV}\frac{\omega}{\alpha},$$

or

$$\mathrm{l}\Pi q = \Sigma\mathrm{lT}q + \angle\,\Pi q.\mathrm{UV}\,\Pi q. \tag{28}$$

The order of the factors must be carefully observed in using this formula.

12°. Let us consider the logarithm of the product of a series of complanar vectors.

First, suppose the number of factors to be even. Pair the successive vectors, and regard these pairs as quaternions. Since they are complanar quaternions, we have, by (20), (20)′, and (27),

$$\mathrm{l}\,(\alpha_1\alpha_2.\alpha_3\alpha_4.\alpha_5\alpha_6 \ldots \alpha_{2n-1}\alpha_{2n}) =$$

$$\sum_1^{2n}\mathrm{lT}\alpha_k + \left[(n - 2m)\partial - \sum_1^{n}\angle\frac{\alpha_{2k}}{\alpha_{2k-1}}\right]\mathrm{UV}\alpha_1\alpha_2, \tag{29}$$

where $(n - 2m + 1)\partial > \left(\Sigma\angle\frac{\alpha_{2k}}{\alpha_{2k-1}}\right) > (n - 2m - 1)\partial,$

and where the angles are to be taken positively or negatively according as they agree or disagree with $\frac{\alpha_2}{\alpha_1}$.

Secondly, if the number of factors be odd, then regard their product, down to and excluding the last vector α_{2n}, as the product of a given number of complanar quaternions. This product, it will be observed, is a quaternion q whose plane is the same as that of the vector factors. The remaining odd vector lies in the plane of q; so that $q\alpha_{2n}$ is a vector lying in the plane of q. Therefore the product of any odd number of complanar vectors is a vector complanar with those vectors. Hence, if $\alpha_0\alpha_1\alpha_2 \ldots \alpha_{2n}$ be the given vectors,

$$\mathrm{l}\,(\alpha_0\alpha_1\alpha_2 \ldots \alpha_{2n}) = \sum_0^{2n}\mathrm{lT}\alpha_k + \frac{\partial}{2}\,\mathrm{U}\prod_0^{2n}\alpha_k. \tag{30}$$

a formula true for any odd number of complanar vectors.

II. APPLICATIONS OF QUATERNION ANALYSIS TO RECTIFICATION OF CURVES, QUADRATURE OF SURFACES, AND CUBATURE OF SOLIDS.

1°. If $\varrho = \varphi\ (t)$ be the equation of any curve in space, it is easily shown that the complete derivative of ϱ, relatively to the scalar variable involved, is the tangent to the curve; —

$$D_t\varrho = \varrho' = \varphi'\ (t). \tag{1}$$

Here $T\varrho'$ is the derivative, and $T\varrho' dt$ the element, of the arc of the curve. Hence, any length of arc will be

$$s - s_0 = \int_{t_0}^{t} T\varrho'. \tag{2}$$

2°. Again, the element of double area swept by the radius vector will be $TV\varrho\varrho' dt$, and any finite area swept by ϱ will be

$$A = \tfrac{1}{2}\int_{t_0}^{t} TV\varrho\varrho'. \tag{3}$$

If the surface be plane, we may change the origin of vectors to a point in the plane, and so find an area measured from the new origin and limited by the limiting positions of the new generating vector.

Let δ be the vector of the new origin (from any origin whatever in space). Then the generating vector will have the form

$$\varpi = \varrho - \delta,$$

and the finite area will be

$$A = \tfrac{1}{2}\int_{t_0}^{t} TV\varpi\varpi' = \tfrac{1}{2}\int_{t_0}^{t} TV(\varrho - \delta)\varrho'. \tag{4}$$

3°. If a plane curve be revolved about a given axis lying in its plane, the surface generated will be one of revolution. If $\varrho = \varphi(t)$ be the vector equation of the curve, referred to a point in the axis of revolution, $T\varrho' dt$ will be the element of arc of the meridian (or generating curve). Let u be the distance of a point of the curve from the axis of revolution, and φ the angle of revolution. Then $ud\varphi$ will be the element of the parallel of latitude, and $ud\varphi T\varrho' dt$ the element of area. Hence, if α is a unit-vector parallel to the axis, the area of a portion of a zone will be

$$S = \Phi \int_{t_0}^{t} u\mathrm{T}\varrho' = \Phi \int_{t_0}^{t} \mathrm{T}.\varrho'\mathrm{V}\alpha\varrho. \tag{5}$$

4°. The direction of the normal to a surface of revolution is the same as that of the perpendicular from the origin (which we still suppose to be in the axis) dropped on the tangent to the meridian. If ϱ' be the tangent to the meridian, the projection of the radius vector ϱ on that tangent is

$$\frac{1}{\rho'}\ \mathrm{S}\varrho\varrho',\ \text{and its length}\ \frac{\mathrm{TS}\rho\rho'}{\mathrm{T}\rho'}.$$

Represent the perpendicular from the origin on the tangent by ν. Then if $\mathrm{T}\nu$ be the length intercepted by the tangent,

$$\begin{aligned}\mathrm{T}\nu^2 &= \mathrm{T}^2\varrho - \frac{\mathrm{S}^2\rho\rho'}{\mathrm{T}^2\rho'},\\ &= \frac{\mathrm{T}^2\rho\rho' - \mathrm{S}^2\rho\rho'}{\mathrm{T}^2\rho'} = \frac{-\mathrm{V}^2\rho\rho'}{\mathrm{T}^2\rho'} = \frac{\mathrm{T}^2\mathrm{V}\rho\rho'}{\mathrm{T}^2\rho'},\end{aligned}$$

and

$$\mathrm{T}\nu = \frac{\mathrm{TV}\rho\rho'}{\mathrm{T}\rho'}. \tag{6}$$

This expression gives the length of the perpendicular from the origin on the tangent plane, and hence the element of volume swept by the radius vector is given by

$$\tfrac{1}{3}\ \mathrm{T}\nu.u\mathrm{d}\varphi\mathrm{T}\varrho'\mathrm{d}t = \tfrac{1}{3}\ \mathrm{TV}\varrho\varrho'.u\mathrm{d}\varphi\mathrm{d}t.$$

The finite volume will therefore be expressed by

$$V = \tfrac{1}{3}\ \Phi \int_{t_0}^{t} u.\mathrm{TV}\varrho\varrho'. \tag{7}$$

5°. Write $\varrho = \varphi(t, u)$ as the equation of a surface, with the condition that t and u are two scalar indeterminates, the changes in t determining a series of successive curves on the surface, these curves intersected by another series of successive curves determined by the changes in u. Then

$$\mathrm{D}_t\varrho = \varrho'_1$$

will be the tangent to a curve of the first series, and

$$\mathrm{D}_u\varrho = \varrho'_2$$

will be the tangent to a curve of the second series. The element of area of the surface will therefore be $\mathrm{TV}\varrho'_1\varrho'_2\mathrm{d}u\mathrm{d}t$, and the finite area is

$$S = \int_u \int_t \mathrm{TV}\varrho'_1\varrho'_2. \tag{8}$$

6°. The direction of the normal to the surface is given by $V\varrho'_1\varrho'_2$, and the projection of the radius vector on the normal is

$$\nu = \frac{1}{V\rho'_1\rho'_2} S\varrho V\varrho'_1\varrho'_2 = \frac{1}{V\rho'_1\rho'_2} S\varrho\varrho'_1\varrho'_2, \qquad (9)$$

and this is the perpendicular from the origin on the tangent plane to the surface at the extremity of ϱ. Its length is

$$T\nu = \pm \frac{S\rho\rho'_1\rho'_2}{TV\rho'_1\rho'_2}. \qquad (10)$$

Hence, the element of volume swept by the radius vector is

$$\tfrac{1}{3}\, TV\varrho'_1\varrho'_2 \cdot \frac{S\rho\rho'_1\rho'_2}{TV\rho'_1\rho'_2}\, du dt,$$

and the finite volume is

$$V = \tfrac{1}{3} \int_u \int_t S\varrho\varrho'_1\varrho'_2. \qquad (11)$$

Here, again, as in 2°, a change of origin will affect the result, and give a different portion of the volume. As before, suppose δ to be the vector of the new origin. Then the new vector, whose extremity generates the surface, is $\varpi = \varrho - \delta$, and the volume swept by ϖ is

$$V = \tfrac{1}{3} \int_u \int_t S\varpi\varpi'_1\varpi'_2 = \tfrac{1}{3} \int_u \int_t S(\varrho - \delta)\varrho'_1\varrho'_2. \qquad (12)$$

7°. The equation of the ellipse may be written

$$\varrho = \alpha \cos x + \beta \sin x, \qquad (13)$$

where α and β are the principal semi-diameters, and x is the eccentric angle. $T\alpha = a$, $T\beta = b$, and α is perpendicular to β. By differentiation

$$D_x\varrho = \varrho' = -\alpha \sin x + \beta \cos x,$$

and taking the tensor

$$T\varrho' = \sqrt{a^2 \sin^2 x + b^2 \cos^2 x}$$

$$= \sqrt{(a^2 - b^2) \sin^2 x + b^2}.$$

Hence the arc of the ellipse is

$$s - s_0 = \int_{x_0}^{x} T\varrho' = \int_{x_0}^{x} \sqrt{(a^2 - b^2) \sin^2 x + b^2}. \qquad (14)$$

The integration of this function, of course, involves elliptic functions For the circle $a = b$, and

$$s - s_0 = \int_{x_0}^{x} b = b(x - x_0).$$

8°. The equation of the hyperbola may be written, x being a variable analogous to the eccentric angle,

$$\varrho = \alpha \operatorname{Ch} x + \beta \operatorname{Sh} x. \tag{15}$$

By differentiation

$$\varrho' = \alpha \operatorname{Sh} x + \beta \operatorname{Ch} x,$$

$$\mathrm{T}\varrho' = \sqrt{a^2 \operatorname{Sh}^2 x + b^2 \operatorname{Ch}^2 x}$$

$$= \sqrt{(a^2 + b^2) \operatorname{Sh}^2 x + b^2}.$$

Hence the expression for the length of the arc is

$$s - s_0 = \int_{x_0}^{x} \sqrt{(a^2 + b^2) \operatorname{Sh}^2 x + b^2}. \tag{16}$$

In the equilateral hyperbola $a = b$, and

$$s - s_0 = \int_{x_0}^{x} a \sqrt{2 \operatorname{Sh}^2 x + 1} = a \int_{x_0}^{x} \sqrt{\operatorname{Ch} 2x}.$$

The equation of the hyperbola referred to its asymptotes is

$$\varrho = x\alpha + x^{-1}\beta. \tag{17}$$

Whence by differentiation

$$\varrho' = \alpha - x^{-2}\beta.$$

Let $\mathrm{T}\alpha = k = \mathrm{T}\beta$. Then

$$\mathrm{T}\varrho' = k \sqrt{1 - 2x^{-2}\mathrm{SU}\alpha\beta + x^{-4}},$$

and

$$s - s_0 = k \int_{x_0}^{x} \frac{\sqrt{x^4 + 2x^2 \cos \frac{\beta}{\alpha} + 1}}{x^2}. \tag{18}$$

In the equilateral hyperbola, $\cos \frac{\beta}{\alpha} = 0$, and

$$s - s_0 = k \int_{x_0}^{x} \frac{\sqrt{x^4 + 1}}{x^2}.$$

9°. The equation of the parabola may be written

$$\rho = \frac{x^2}{2}\alpha + x\beta, \tag{19}$$

where $\alpha \perp \beta$. Then

$$\rho' = x\alpha + \beta,$$

and if $T\alpha = a$, and $T\beta = b$,

$$T\rho' = \sqrt{a^2x^2 + b^2},$$

and

$$s - s_0 = \int_{x_0}^{x} \sqrt{a^2x^2 + b^2}$$

$$= \left[\frac{x}{2}\sqrt{a^2x^2 + b^2} + \frac{b^2}{2a}\,\mathrm{Sh}^{-1}\frac{ax}{b}\right]_{x_0}^{x}$$

$$= \frac{b^2}{2a}\left[\frac{ax}{b}\sqrt{\frac{a^2x^2}{b^2} + 1} + \mathrm{Sh}^{-1}\frac{ax}{b}\right]_{x_0}^{x}.$$

Let $v = \mathrm{Sh}^{-1}\frac{ax}{b}$; then the expression for the arc becomes

$$s - s_0 = \frac{b^2}{4a}\left[\mathrm{Sh}\ 2v + 2v\right]_{v_0}^{v}. \tag{20}$$

10°. The equation of the helix on the elliptic cylinder (or, rather, of a curve analogous to the helix) is

$$\rho = \alpha\cos x + \beta\sin x + \gamma x, \tag{21}$$

where $\alpha \perp \beta \perp \gamma \perp \alpha$, α and β being semi-diameters of the elliptic base. By differentiation,

$$\rho' = -\alpha\sin x + \beta\cos x + \gamma;$$

$$T\rho' = \sqrt{(a^2\sin^2 x + b^2\cos^2 x + c^2)},$$

if $T\alpha = a$, $T\beta = b$, $T\gamma = c$.

$$\therefore\ s - s_0 = \int_{x_0}^{x} \sqrt{(a^2 - b^2)\sin^2 x + b^2 + c^2}. \tag{22}$$

This integration, as it should, involves elliptic functions. If the base of the cylinder be circular, then $a = b$, and

$$s - s_0 = \int_{x_0}^{x} \sqrt{b^2 + c^2} = \sqrt{b^2 + c^2}(x - x_0) \tag{23}$$

11°. The scalar equations of the cycloid are

$$x = a\ (\theta - \sin\theta),$$
$$y = a\ (1 - \cos\theta).$$

Let $T\alpha = T\beta = a$. Then the vector equation of the cycloid may be written

$$\varrho = xU\alpha + yU\beta = \alpha(\theta - \sin\theta) + \beta(1 - \cos\theta). \quad (24)$$

Whence

$$\varrho' = \alpha(1 - \cos\theta) + \beta\sin\theta,$$
$$\varrho'^2 = \alpha^2(1 - \cos\theta)^2 + \beta^2\sin^2\theta,$$
$$T\varrho' = a\sqrt{1 - 2\cos\theta + \sin^2\theta + \cos^2\theta}$$
$$= a\sqrt{2 - 2\cos\theta} = 2a\sin\tfrac{1}{2}\theta.$$
$$\therefore\quad s - s_0 = 2a\int_{\theta_0}^{\theta}\sin\tfrac{1}{2}\theta = 4a\Big[\cos\tfrac{1}{2}\theta\Big]_{\theta}^{\theta_0}. \quad (25)$$

For the complete arc

$$s - s_0 = 4a\Big[\cos\tfrac{1}{2}\theta\Big]_{2\circlearrowleft}^{0} = 8a.$$

12°. The areas of the conic sections are easily obtained from formulas (3) and (4). For the ellipse we have, α and β being any two conjugate semi-diameters,

$$\varrho = \alpha\cos x + \beta\sin x,$$
$$\varrho' = -\alpha\sin x + \beta\cos x,$$

whence

$$TV\varrho\varrho' = TV(\alpha\beta\cos^2 x - \beta\alpha\sin^2 x) = TV\alpha\beta;$$
$$\therefore\quad A = \tfrac{1}{2}\int_{x_0}^{x} TV\varrho\varrho' = \frac{TV\alpha\beta}{2}\Big[x - x_0\Big]$$
$$= \frac{ab}{2}\sin{}^{\beta}_{\alpha}\Big[x - x_0\Big]. \quad (26)$$

Or, if $\alpha \perp \beta$,

$$A = \frac{ab}{2}\Big[x - x_0\Big].$$

The whole area is $\circlearrowleft ab$.

We may change the origin to any fixed point, and so obtain the area of any portion of the ellipse. Suppose ε is the vector of the fixed point, and ϖ the new radius vector from this point; then the equation of the ellipse becomes

$$\varpi = (\varrho - \varepsilon) = \alpha\cos x + \beta\sin x - \varepsilon.$$

Whence

$$\varpi' = -\alpha \sin x + \beta \cos x = \rho',$$

$$\mathrm{TV}\varpi\varpi' = \mathrm{TV}\,(\alpha\beta + \varepsilon\alpha \sin x - \varepsilon\beta \cos x),$$

and

$$A = \tfrac{1}{2}\int_{x_0}^{x} \mathrm{TV}(\alpha\beta + \varepsilon\alpha \sin x - \varepsilon\beta \cos x). \qquad (27)$$

If the new origin lie anywhere on α or β, one of the terms of this integration disappears. Suppose the new origin is on α; then $\varepsilon = m\alpha$, where m is a scalar. Hence

$$A = \tfrac{1}{2}\int_{x_0}^{x} \mathrm{TV}\alpha\beta(1 - m \cos x)$$

$$= \frac{ab}{2} \sin \frac{\beta}{\alpha} \Big[x - m \sin x\Big]_{x_0}^{x}. \qquad (28)$$

13°. The equation of the hyperbola,

$$\rho = \alpha \mathrm{Ch}x + \beta \mathrm{Sh}x,$$

will give results precisely similar to those of the preceding section, with the hyperbolic sine and cosine everywhere substituted for the circular. With the origin of vectors at the centre, the area swept by the radius vector is

$$A = \frac{ab}{2} \sin \frac{\beta}{\alpha} \Big[x - x_0\Big]; \qquad (29)$$

or if $\alpha \perp \beta$

$$A = \frac{ab}{2} \Big[x - x_0\Big].$$

This is the area of a portion of the surface exterior to the curve, contained between the curve and the limiting positions of the radius vector. To find a portion of the inner area we need only transfer the origin to some point on or within the curve, and proceed as before. Suppose the origin to lie on α, at a distance $m\mathrm{T}\alpha$ from the centre. The finite area is

$$A = \frac{ab}{2} \sin \frac{\beta}{\alpha} \Big[x - m\mathrm{Sh}x\Big]_{x_0}^{x}. \qquad (30)$$

In formulae (28) and (30), put $m = \cos x_m$ and $m = \mathrm{Ch}x_m$, and take x_m and 0 for the limits. Then if $\alpha \perp \beta$, the elliptic area (doubled) is

$$A_1 = ab[x_m - \cos x_m \sin x_m], \qquad (31)$$

and the hyperbolic area (doubled) is

$$A_2 = ab[x_m - \mathrm{Ch}x_m\mathrm{Sh}x_m]. \qquad (32)$$

These areas are those of segments cut by lines parallel to β.

14°. In the equation of the parabola

$$\varrho = \frac{x^2}{2}\alpha + x\beta,$$

let $\mathrm{T}\alpha = a$, $\mathrm{T}\beta = b$. In this equation of the curve, α is the diameter and β is the tangent at the origin. We have

$$\varrho' = x\alpha + \beta,$$

and

$$\mathrm{TV}\varrho\varrho' = \mathrm{TV}(\frac{x^2}{2}\alpha\beta - x^2\alpha\beta) = \mathrm{TV}\alpha\beta\,\frac{x^2}{2},$$

$$A = \frac{ab}{4}\sin{}^{\beta}_{\alpha}\int_{x_0}^{x} x^2 = \frac{ab\sin{}^{\beta}_{\alpha}}{12}\,[x^3 - x_0^{\,3}]; \qquad (33)$$

this being the area of a sector of which the vertex is at the origin. If $\alpha \perp \beta$, then

$$A = \frac{ab}{12}\left[x^3 - x_0^{\,3}\right].$$

Transfer the origin to any point on the tangent β. Let

$$\varepsilon = m\beta,$$

where m is a scalar. The new equation becomes

$$(\varrho - \varepsilon) = \varpi = \frac{x^2}{2}\alpha + (x - m)\beta.$$

Whence

$$\varpi' = x\alpha + \beta,$$

$$\mathrm{TV}\varpi\varpi' = \mathrm{TV}\left[\frac{x^2}{2}\alpha\beta - (x^2 - xm)\alpha\beta\right]$$

$$= \mathrm{TV}\alpha\beta\left(xm - \frac{x^2}{2}\right),$$

and

$$A = \frac{ab\sin{}^{\beta}_{\alpha}}{2}\int_{x_0}^{x}(xm - \frac{x^2}{2})$$

$$= \frac{ab\sin{}^{\beta}_{\alpha}}{2}\left[\frac{x^2m}{2} - \frac{x^3}{6}\right]_{x_0}^{x}. \qquad (34)$$

If the sector be cut with one boundary parallel to α, so that at the upper limit $x = m$, while x_0 is taken $= 0$, then

$$A = \frac{ab}{2} \sin\frac{\beta}{\alpha} \left[\frac{x^3}{2} - \frac{x^3}{6}\right] = \frac{abx^3}{6} \sin\frac{\beta}{\alpha}. \qquad (35)$$

This is one-third of the parallelogram formed by the co-ordinates of the point at the extremity of ϱ.

15°. The equation of the cycloid gives the following:—

$$\varrho = \alpha(\theta - \sin\theta) + \beta(1 - \cos\theta),$$

$$\varrho' = \alpha(1 - \cos\theta) + \beta\sin\theta,$$

$$\mathrm{TV}\varrho\varrho' = \mathrm{TV}[\alpha\beta(\theta\sin\theta - \sin^2\theta - \{1 - \cos\theta\}^2)]$$

$$= a^2(\theta\sin\theta - 4\sin^2\tfrac{1}{2}\theta).$$

Hence

$$A = \frac{a^2}{2}\int_{\theta_0}^{\theta}(\theta\sin\theta + 2\cos\theta - 2)$$

$$= \frac{a^2}{2}\Big[\sin\theta - \theta\cos\theta + 2\sin\theta - 2\theta\Big]_{\theta_0}^{\theta}$$

$$= \frac{a^2}{2}\Big[3\sin\theta - \theta\cos\theta - 2\,\theta\Big]_{\theta_0}^{\theta}. \qquad (36)$$

The complete area is $A = 3\circlearrowleft a^2$.

16°. In 14°, we found for the parabola

$$\varrho = \frac{x^2}{2}\alpha + x\beta,$$

$$\mathrm{T}\varrho' = \sqrt{a^2x^2 + b^2}.$$

Revolve the curve about β, the tangent at the vertex. Make

$$u = \frac{ax^2}{2}.$$

The area of the surface thus generated will be [see (5)]

$$S = \Phi\int_{x_0}^{x} u\mathrm{T}\varrho' = \Phi\,\frac{a}{2}\int_{x_0}^{x} x^2\sqrt{a^2x^2 + b^2}$$

$$= \Phi\left[\frac{(2\,a^2x^2 + b^2)x\sqrt{a^2x^2 + b^2}}{16a} - \frac{b^4}{16a^2}\,\mathrm{Sh}^{-1}\frac{ax}{b}\right]_{x_0}^{x}$$

$$= \Phi \, \frac{b^4}{64a^2}\Big[\mathrm{Sh}4v - 4v\Big]_{v_0}^{v}, \qquad (37)$$

where $v = \mathrm{Sh}^{-1}\frac{ax}{b}$.

If we revolve the curve about the diameter α, we may then put $u = bx$, and the area of the surface generated will be found to be

$$S = \Phi b \int_{x_0}^{x} x\sqrt{a^2x^2 + b^2} = \Phi b \Big[\frac{(a^2x^2 + b^2)^{\frac{3}{2}}}{3\,a^2}\Big]_{x_0}^{x}$$

$$= \Phi \, \frac{b^4}{3a^2}\Big[\mathrm{Ch}^3 v\Big]_{v_0}^{v}, \qquad (38)$$

where $v = \mathrm{Sh}^{-1}\frac{ax}{b}$.

17°. Again, in 14°, we found for the parabola, when $\alpha \perp \beta$,

$$\mathrm{TV}\rho\rho' = \frac{ab}{2}\, x^2.$$

Whence, for the volume of the solid of revolution (generated by the revolution of a sector), the axis being the tangent at the vertex, we have, making $u = \frac{a}{2}\, x^2$,

$$V = \tfrac{1}{3}\, \Phi \int_{x_0}^{x} u.\mathrm{TV}\rho\rho' = \Phi \, \frac{a^2b}{12}\int_{x_0}^{x} x^4 = \Phi \, \frac{a^2b}{60}\Big[x^5 - x_0^{\,5}\Big]; \qquad (39)$$

and for the volume of the paraboloid whose axis is the diameter, making $u = bx$,

$$V = \Phi \, \frac{ab^2}{6}\int_{x_0}^{x} x^3 = \Phi \, \frac{ab^2}{24}\Big[x^4 - x_0^{\,4}\Big]. \qquad (40)$$

18°. For the cycloid we have found, in 11°,

$$\rho = \alpha(\theta - \sin\theta) + \beta(1 - \cos\theta),$$

$$\mathrm{T}\rho' = 2a\,\sin\tfrac{1}{2}\theta.$$

If we revolve it about the base, we shall have $u = a(1 - \cos\theta)$, and the area of the surface generated will be

$$S = \Phi \int_{\theta_0}^{\theta} u\mathrm{T}\rho' = 2\,\Phi a^2 \int_{\theta_0}^{\theta} \sin\tfrac{1}{2}\,\theta\,(1 - \cos\theta)$$

$$= 8\ \Phi a^2 \int_{\frac{1}{2}\theta_0}^{\frac{1}{2}\theta} \sin^3 \tfrac{1}{2}\,\theta = 8\Phi a^2 \Big[\tfrac{1}{3} \cos^3 \tfrac{1}{2}\,\theta - \cos \tfrac{1}{2}\,\theta\Big]_{\theta_0}^{\theta}. \quad (41)$$

The complete area is $S = \frac{64}{3}\,\varpi a^2$.

The area of the surface formed by revolving the cycloid about β would be found by putting $u = a(\theta - \sin\theta)$ and using formula (5) as before.

19°. The volume generated by revolving a sector of the cycloid about its base is

$$V = \frac{\Phi}{3} \int_{\theta_0}^{\theta} u.\mathrm{TV}\varrho\varrho' = \Phi\,\frac{a^3}{3} \int_{\theta_0}^{\theta} (1 - \cos\theta)(\theta \sin\theta + 2\cos\theta - 2)$$

$$= \Phi\,\frac{a^3}{3} \int_{\theta_0}^{\theta} (\theta \sin\theta + 4\cos\theta - \tfrac{1}{2}\,\theta \sin 2\,\theta - 2\cos^2\theta - 2)$$

$$= \Phi\,\frac{a^3}{3} \Big[5 \sin\theta - \frac{5}{8} \sin 2\,\theta - \theta(\cos\theta - \tfrac{1}{4} \cos 2\,\theta + 3)\Big]_{\theta_0}^{\theta}. \quad (42)$$

The complete volume is $5\ \varpi^2 a^3$.

20°. In 7°, we have found for the ellipse

$$\mathrm{T}\varrho' = \sqrt{(a^2 - b^2)\sin^2 x + b^2} = \sqrt{a^2 - (a^2 - b^2)\cos^2 x}.$$

The prolate ellipsoid is generated by revolving the ellipse about its major axis. In this case, the equation being $\varrho = \alpha \cos x + \beta \sin x$, we shall have

$$u = b \sin x,$$

and for the surface

$$S = \Phi \int_{x_0}^{x} u.\mathrm{T}\varrho' = \Phi b \int_{x_0}^{x} \sin x \sqrt{a^2 - (a^2 - b^2)\cos^2 x}$$

$$= \Phi b \int_{\cos x_0}^{\cos x} \sqrt{a^2 - c^2 \cos^2 x},$$

where $c^2 = a^2 - b^2$. Whence

$$S = \Phi b \Big[\frac{\cos x}{2} \sqrt{a^2 - c^2 \cos^2 x} + \frac{a^2}{2c} \sin^{-1} \frac{c \cos x}{a}\Big]_{x_0}^{x}$$

$$= \Phi\,\frac{a^2b}{2c}\left[\frac{c\cos x}{a}\sqrt{1-\frac{c^2\cos^2 x}{a^2}}+\sin^{-1}\frac{c\cos x}{a}\right]_{x_0}^{x}.$$

Let $v = \sin^{-1}\frac{c\cos x}{a}$; then

$$S = \Phi\,\frac{a^2b}{4c}\Big[\sin 2v + 2v\Big]_{v_0}^{v}. \qquad (43)$$

For the oblate ellipsoid of revolution, the substitution

$$u = a\cos x$$

gives, by the same method of proceeding as before,

$$S = \Phi a\int_{x_0}^{v}\cos x\sqrt{c^2\sin^2 x+b^2} = \Phi a\int_{\sin x_0}^{\sin x}\sqrt{c^2\sin^2 x+b^2}$$

$$= \Phi a\left[\frac{\sin x}{2}\sqrt{c^2\sin^2 x+b^2}+\frac{b^2}{2c}\,\mathrm{Sh}^{-1}\frac{c\sin x}{b}\right]_{x_0}^{x}$$

$$= \Phi\,\frac{ab^2}{2c}\left[\frac{c\sin x}{b}\sqrt{\frac{c^2\sin^2 x}{b^2}+1}+\mathrm{Sh}^{-1}\frac{c\sin x}{b}\right]_{x_0}^{x}.$$

Let $v = \mathrm{Sh}^{-1}\frac{c\sin x}{b}$; then

$$S = \Phi\,\frac{ab^2}{4c}\Big[\mathrm{Sh}\,2v + 2v\Big]_{v_0}^{v}. \qquad (44)$$

21°. From the equation of the hyperbola we found (8°),

$$\mathrm{T}\varrho' = \sqrt{c^2\mathrm{Sh}^2x+b^2} = \sqrt{c^2\mathrm{Ch}^2x-a^2},$$

where $c^2 = a^2 + b^2$. Hence, for the unparted hyperboloid of revolution, generated by revolving the hyperbola about the axis β, we put

$$u = a\mathrm{Ch}x,$$

and find

$$S = \Phi\int_{x_0}^{x} u.\mathrm{T}\varrho' = \Phi a\int_{x_0}^{x}\mathrm{Ch}x\sqrt{c^2\mathrm{Sh}^2x+b^2}$$

$$= \Phi\,\frac{ab^2}{2c}\left[\frac{c\mathrm{Sh}x}{b}\sqrt{\frac{c^2\mathrm{Sh}^2x}{b^2}+1}+\mathrm{Sh}^{-1}\frac{c\mathrm{Sh}x}{b}\right]_{x_0}^{x}$$

$$= \Phi\,\frac{ab^2}{4c}\Big[\mathrm{Sh}2v + 2v\Big]_{v_0}^{v}, \qquad (45)$$

where $v = \mathrm{Sh}^{-1}\frac{c\mathrm{Sh}x}{b}$.

Again, the substitution $u = b\mathrm{Sh}x$ gives, for the area of the parted hyperboloid of revolution,

$$S = \Phi b \int_{x_0}^{x} \mathrm{Sh}x\sqrt{c^2\mathrm{Ch}^2x - a^2}$$

$$= \Phi\,\frac{a^2b}{2c}\left[\frac{c\mathrm{Ch}x}{a}\sqrt{\frac{c^2\mathrm{Ch}^2x}{a^2} - 1} - \mathrm{Ch}^{-1}\,\frac{c\mathrm{Ch}x}{a}\right]_{x_0}^{x}$$

$$= \Phi\,\frac{a^2b}{4c}\Big[\mathrm{Sh}\;2v - 2v\Big]_{v_0}^{v}, \qquad (46)$$

where $v = \mathrm{Ch}^{-1}\,\dfrac{c\mathrm{Ch}x}{a}$.

22°. By the application of formula (7) the volume generated by the revolution of a sector of a conic about a principal axis may be obtained very simply. With $\alpha \perp \beta$, the equations of the ellipse and hyperbola give

$$\mathrm{TV}\varrho\varrho' = \mathrm{TV}\alpha\beta = ab.$$

The substitution $u = b \sin x$ gives

$$V = \tfrac{1}{3}\,\Phi \int u\mathrm{TV}\varrho\varrho' = \tfrac{1}{3}\,\Phi\; ab^2 \int_{x_0}^{x} \sin x$$

$$= \tfrac{1}{3}\,\Phi\; ab^2\,\Big[\cos x\Big]_{x}^{x_0}, \qquad (47)$$

for the volume enclosed by the prolate ellipsoid; and the substitution $u = a \cos x$, gives

$$V = \tfrac{1}{3}\,\Phi\; a^2b \int_{x_0}^{x} \cos x = \tfrac{1}{3}\,\Phi\; a^2b\Big[\sin x\Big]_{x_0}^{x}, \qquad (47')$$

for the volume enclosed by the oblate ellipsoid.

23°. Similarly, the substitution $u = a\mathrm{Ch}x$ gives

$$V = \Phi\,\frac{a^2b}{3}\int_{x_0}^{x} \mathrm{Ch}x = \Phi\,\frac{a^2b}{3}\Big[\mathrm{Sh}x\Big]_{x_0}^{x}, \qquad (48)$$

for the volume swept by a central sector in generating the unparted hyperboloid; and the substitution $u = b\mathrm{Sh}x$ gives

$$V = \Phi\,\frac{ab^2}{3}\int_{x_0}^{x} \mathrm{Sh}x = \Phi\,\frac{ab^2}{3}\Big[\mathrm{Ch}x\Big]_{x_0}^{x}, \qquad (48')$$

for the volume swept by a central sector in generating the parted hyperboloid.

The volumes enclosed by the surfaces of the hyperboloids will be considered later in connection with the discussion of the general equations of the quadric surfaces.

To obtain the volumes bounded by the general ellipsoids and elliptic hyperboloids, it is only necessary, in the results of 22° and 23°, by a well-known principle, to change a^2 into ac, or b^2 into bc, where c is the length of the third semi-diameter.

24°. The general discussion of the areas and volumes of a very important class of surfaces will be facilitated by writing, in each case, a quaternion equation of the surface in terms of two independent scalar variables.

Let α, β, γ — three vectors diverging from the origin — be any three axes of the surface whose generating vector is ρ. Let $\sigma =$ the vector — complanar with ρ and α — whose extremity describes the section of the surface in the plane of $\beta\gamma$. Write the equation of this section in the form

$$\sigma = \beta f_3 y + \gamma f_4 y,$$

and the equation of the surface in the form

$$\rho = \alpha f_1 x + \sigma f_2 x, \qquad (49)$$

where f_1, f_2, etc., are separately functions of a single variable. In this equation (49), the two scalar variables are evidently independent of each other, a new value of either determining a new point on the surface; ρ will describe, if x remain constant, a section parallel to the plane of $\beta\gamma$, if y remain constant, a section whose plane contains α. For convenience we may suppose $\alpha \perp \beta \perp \gamma$. We shall now have

$D_x\rho = \rho'_1 =$ tangent to meridian at extremity of ρ,

$D_y\rho = \rho'_2 =$ tangent to parallel of latitude at extremity of ρ; and, performing the differentiation,

$$\rho'_1 = \alpha f_1' x + \sigma f_2' x, \quad \rho'_2 = \sigma' f_2 x,$$

whence

$$V\rho'_1\rho'_2 = V\alpha\sigma' f_2 x f_1' x + V\sigma\sigma' f_2 x f_2' x,$$

and, since $V\alpha\sigma'$ and $V\sigma\sigma'$ are perpendicular to each other,

$$V^2\rho'_1\rho'_2 = (f_2 x)^2[V^2\alpha\sigma'(f_1'x)^2 + V^2\sigma\sigma'(f_2'x)^2],$$

or

$$TV\rho'_1\rho'_2 = f_2 x \sqrt{[T^2\alpha\sigma'(f_1'x)^2 + T^2V\sigma\sigma'(f_2'x)^2]}. \qquad (50)$$

This result, substituted in formula (8), gives

$$S = \int_{y}\int_{x} f_2x \sqrt{[\mathrm{T}^2\alpha\sigma'(f_1'x)^2 + \mathrm{T}^2\mathrm{V}\sigma\sigma'(f_2'x)^2]}. \qquad (51)$$

Let us now write a system of equations for the quadric surfaces. For this purpose we have only to substitute, in equation (49), such functions of x and of y as shall cause σ, when y varies, and ρ, when y is constant and x varies, to describe conic sections; and any functions which identically satisfy the conditions $(f_1x)^2 \pm (f_2x)^2 = 1$, $(f_3x)^2 \pm (f_4x)^2 = 1$, or any equation of the second degree in f_1x, $(f_2x.f_3y)$, and $(f_2x.f_4y)$, will evidently serve our purpose. We may write then, for a system of quadric surfaces, the following equations: —

I. $\rho = \alpha \cos x + \sigma \sin x$,
$\sigma = \beta \cos y + \gamma \sin y$; the ellipsoid;

II. $\rho = \alpha \mathrm{Ch}x + \sigma \mathrm{Sh}x$,
$\sigma = \beta \cos y + \gamma \sin y$; the parted hyperboloid, having α for its principal axis;

III. $\rho = \alpha \mathrm{Sh}x + \sigma \mathrm{Ch}x$,
$\sigma = \beta \cos y + \gamma \sin y$; the unparted hyperboloid, having α for its principal axis;

IV. $\rho = \alpha \frac{x^2}{2} + \sigma x$,
$\sigma = \beta \cos y + \gamma \sin y$; the elliptic paraboloid, having α for its axis;

V. $\rho = \alpha \frac{x^2}{2} + \sigma x$,
$\sigma = \beta \mathrm{Ch}y + \gamma \mathrm{Sh}y$; the hyperbolic paraboloid, having α for its axis;

and finally I write

VI. $\rho = \alpha x + \sigma \frac{x^2}{2}$,
$\sigma = \beta \cos y + \gamma \sin y$, as being analogous to IV. and V., though not representing a quadric.

The function $\mathrm{TV}\rho'_1\rho'_2$ derived from I., — if we observe that $\mathrm{V}\sigma\sigma' = \beta\gamma$, and make the substitutions $f_1x = \cos x$, $f_1'x = - \sin x$, etc., in (50), — now becomes

$$\mathrm{TV}\rho'_1\rho'_2 = \sin x \sqrt{(a^2s^2 \sin^2 x + b^2c^2 \cos^2 x)}$$

$$= s \sin x \sqrt{(a^2 - \frac{a^2s^2 - b^2c^2}{s^2} \cos^2 x)},$$

where $a = \mathrm{T}\alpha$, $b = \mathrm{T}\beta$, $c = \mathrm{T}\gamma$, $s = \mathrm{T}\sigma'$. Hence, for the general ellipsoid,

$$S = \int_y s \int_x \sin x \sqrt{(a^2 - \frac{a^2s^2 - b^2c^2}{s^2} \cos^2 x)}$$

$$= -\int_y s . \int_{\cos x} \sqrt{(a^2 - \frac{a^2s^2 - b^2c^2}{s^2} \cos^2 x)},$$

and, if $m^2 = \frac{a^2s^2 - b^2c^2}{a^2s^2}$,

$$S = \int_y \frac{sa}{4m}\Big[2m \cos x \sqrt{(1 - m^2 \cos x)} + 2 \sin^{-1}(m \cos x)\Big]_{x_0}^{x},$$

I′. $$\text{or } S = \int_y \frac{sa}{4m}\Big[\sin 2v + 2v\Big]_{v_0}^{v},$$

where $v = \sin^{-1}(m \cos x)$.

If the surface be a prolate ellipsoid of revolution, then $b = c = s$. $m^2 = \frac{a^2 - b^2}{a^2}$, and

$$S = \Big[y - y_0\Big]\frac{ab}{4m}\Big[\sin 2v + 2v\Big]_{v_0}^{v},$$

a result which agrees with that obtained in 20°.

Repeating this process of integration for cases II. and III., we obtain in succession

II′. $$S = \int_y \frac{sa}{4m}\Big[\mathrm{Sh}2v - 2v\Big]_{v_0}^{v},$$

where $v = \mathrm{Ch}^{-1}(m\mathrm{Ch}x)$ and $m^2 = \frac{a^2s^2 + b^2c^2}{a^2s^2}$;

III′. $$S = \int_y \frac{sa}{4m}\Big[\mathrm{Sh}2v + 2v\Big]_{v_0}^{v},$$

where $v = \mathrm{Sh}^{-1}(m\mathrm{Sh}x)$ and $m^2 = \frac{b^2c^2 - a^2s^2}{a^2s^2}$.

For cases IV. and V., we shall have $\rho'_1 = \alpha x + \sigma$, $\rho'_2 = \sigma' x$, and

$$\mathrm{V}\rho'_1\rho'_2 = \alpha\sigma' x^2 + \beta\gamma x,$$

whence

IV′. $$S = \int_y \int_x x \sqrt{(a^2s^2x^2 + b^2c^2)} = \int_y \Big[\frac{(a^2s^2x^2 + b^2c^2)^{\frac{3}{2}}}{3a^2s^2}\Big]_{x_0}^{x}$$

$$=\int_y \frac{b^3c^3}{3a^2s^2}\Big[\mathrm{Ch}^3 v\Big]_{v_0}^{v},$$

where $v = \mathrm{Sh}^{-1}\,\frac{asx}{bc}$;

V′. = IV′. with a different value for s; and finally VI. gives $\varrho'_1 = \sigma x + \alpha$, $\varrho'_2 = \frac{1}{2}\,\sigma' x^2$, whence

$$\mathrm{V}\varrho'_1\varrho'_2 = \beta\gamma\,\frac{x^3}{2} + \alpha\sigma'\,\frac{x^2}{2},$$

and

$$\text{VI}'.\qquad S = \int_y\int_x \frac{x^2}{2}\sqrt{(b^2c^2x^2 + a^2s^2)} = \int_y \frac{a^4s^4}{64b^3c^3}\Big[\mathrm{Sh}4v - 4v\Big]_{v_0}^{v},$$

where $v = \mathrm{Sh}^{-1}\,\frac{bcx}{as}$. The equations for the surface generated by revolving the parabola about its tangent at the vertex, will be found from VI. and VI′. by putting $b = c = s$, and its finite area is

$$S = \Big[y - y_0\Big]\frac{a^4}{64.b^2}\Big[\mathrm{Sh}4v - 4v\Big]_{v_0}^{v},$$

where $v = \mathrm{Sh}^{-1}\,\frac{bx}{a}$.

25°. Various forms of equations may be used for the quadrics. Thus, the parted hyperboloid represented by III. may also be represented by the equations

$$\varrho = \beta\mathrm{Sh}x + \sigma\mathrm{Ch}x,$$

$$\sigma = \alpha\mathrm{Ch}y + \gamma\mathrm{Sh}y.$$

Again, the unparted hyperboloid represented by IV. can also be represented by

$$\varrho = \beta\cos x + \sigma\sin x,$$

$$\sigma = \alpha\mathrm{Sh}y + \gamma\mathrm{Ch}y,$$

or by

$$\varrho = \pm\,\beta\mathrm{Ch}x + \sigma\mathrm{Sh}x,$$

$$\sigma = \alpha\mathrm{Ch}y + \gamma\mathrm{Sh}y.$$

But, with regard to the last two sets of equations, it is to be observed that they must be taken together, in order to give a complete representation of the surface, for real values of the variables x and y. For, if we use only real values of the variables, the first set of equations gives no points of the surface exterior to a pair of planes parallel to the plane of $\alpha\gamma$, and passing through the extremities of $+\beta$ and $-\beta$; whereas the second set gives no points of the surface contained between these planes. These might be called, therefore, *supplemental equations* of the surface.

If the equations be reduced to scalar forms in rectangular coördinates, they become identical, and their limitations disappear. Thus we have: —

$$x_1 = a \sin x \,\mathrm{Sh}y, \qquad x_2 = a \,\mathrm{Sh}x \,\mathrm{Ch}y,$$

$$y_1 = b \cos x, \qquad y_2 = \pm\, b \,\mathrm{Ch}x,$$

$$z_1 = c \sin x \,\mathrm{Ch}y, \qquad z_2 = c \,\mathrm{Sh}x \,\mathrm{Sh}y,$$

and hence

$$-\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 = -\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2}.$$

Imaginary values of the variables x and y in the equations would, however, make either set complete.

26°. Formula (11), for finding volumes, is

$$V = \tfrac{1}{3} \int_y \int_x \mathrm{S}\rho\rho'_1\rho'_2.$$

From the equation of the general ellipsoid, it is at once evident that

$$\mathrm{V}\rho\rho'_1 = \mathrm{V}\alpha\sigma, \; \mathrm{V}\sigma\sigma' = \mathrm{V}\beta\gamma,$$

whence follows

$$\mathrm{S}\rho\rho'_1\rho'_2 = \mathrm{S}\rho'_2\mathrm{V}\rho\rho'_1 = \sin x \,\mathrm{S}\sigma'\mathrm{V}\alpha\sigma$$

$$= \sin x \,\mathrm{S}\alpha\mathrm{V}\sigma\sigma' = \sin x \,\mathrm{S}\alpha\beta\gamma = abc \sin x.$$

The ellipsoidal volume is therefore

$$V = \frac{abc}{3} \int_y \int_x \sin x = \Phi \frac{abc}{3} [\cos x_0 - \cos x]. \qquad (52)$$

The whole volume is $V = \frac{4}{3} \partial abc.$

27°. From the equation of the parted hyperboloid are easily and directly obtained

$$\mathrm{S}\rho\rho'_1\rho'_2 = \mathrm{S}\rho'_2\mathrm{V}\rho\rho'_1 = \mathrm{Sh}x\mathrm{S}\sigma'\mathrm{V}\alpha\sigma$$

$$= \mathrm{Sh}x\mathrm{S}\alpha\mathrm{V}\sigma\sigma' = \mathrm{Sh}x\mathrm{S}\alpha\beta\gamma = abc\mathrm{Sh}x,$$

and

$$V = \frac{abc}{3}\int_y\int_x \mathrm{Sh}x = \Phi\frac{abc}{3}\Big[\mathrm{Ch}x\Big]_{x_0}^{x}. \tag{53}$$

Similarly the equation of the unparted hyperboloid gives

$$S\varrho\varrho'_1\varrho'_2 = -\mathrm{Ch}x S\sigma'\alpha\sigma = \mathrm{Ch}x S\alpha V\sigma'\sigma$$

$$= \mathrm{Ch}x S\alpha\gamma\beta = -abc\mathrm{Ch}x,$$

and

$$V = \frac{abc}{3}\int_y\int_x \mathrm{Ch}x = \Phi\frac{abc}{3}\Big[\mathrm{Sh}x\Big]_{x_0}^{x}. \tag{54}$$

28°. Formulae (52), (53), and (54), it will be observed, give the volume of the portion of space swept by the radius vector; *i.e.*, of the space contained between the surface of the quadric and the two cones whose vertices are at the centre, and which are generated by the limiting positions of the radius vector at the limits of integration; or, if the lower limit is zero, the volume contained between the surface and a single central cone. The volume of the segment cut off by a plane perpendicular to an axis; can be found by finding the difference between the volume given by the general formula and the whole volume of this cone considered as limited by the plane; but more easily and simply by a change of origin, according to the method of 6°, by the use of formula (12).

29°. Beginning with the ellipsoid, let the origin be transferred to a fixed point on α, and let ϖ be the new generating vector. Then

$$\varpi = \varrho - m\alpha,$$

m being a scalar. Formula (12) becomes in this case

$$V = \tfrac{1}{3}\int_y\int_x S\varpi\varrho'_1\varrho'_2 = \tfrac{1}{3}\int_x\int_y S(\varrho - m\alpha)\varrho'_1\varrho'_2.$$

We have, as in 24°,

$$\varrho'_1 = -\alpha\sin x + \sigma\cos x,$$

$$\varrho'_2 = \sigma'\sin x,$$

hence

$$mS\alpha\varrho'_1\varrho'_2 = mS.\alpha(-\alpha\sin x + \sigma\cos x)\sigma'\sin x$$

$$= m\sin x\cos x S\alpha\beta\gamma = mabc\sin x\cos x.$$

In 26°, we found

$$S\varrho\varrho'_1\varrho'_2 = abc\sin x.$$

The ellipsoidal volume will therefore be

$$V = \Phi \frac{abc}{3} \int_x (\sin x - m \sin x \cos x)$$

$$= \Phi \frac{abc}{3} \Big[\cos x + \frac{m}{2} \sin^2 x \Big]_x^{x_0}. \qquad (55)$$

If m be determined by the condition $m = \cos x$, where x is the superior limit of integration, then $\varpi = \sigma \sin x$ at that limit, showing that the volume obtained will be limited by a plane perpendicular to α; and if the lower limit is 0, we have for the volume of a segment: —

$$V = \Phi \frac{abc}{3} \Big[1 - \cos x - \tfrac{1}{2} \cos x \sin^2 x \Big]. \qquad (56)$$

30°. Similarly, the parted hyperboloid gives

$$\varrho'_1 = \alpha \mathrm{Sh} x + \sigma \mathrm{Ch} x,$$

$$\varrho'_2 = \sigma' \mathrm{Sh} x,$$

$$\mathrm{S} \varrho \varrho'_1 \varrho'_2 = abc \mathrm{Sh} x,$$

$$m \mathrm{S} \alpha \varrho'_1 \varrho'_2 = m \mathrm{S}.\alpha (\alpha \mathrm{Sh} x + \sigma \mathrm{Ch} x) \sigma' \mathrm{Sh} x$$

$$= mabc \mathrm{Sh} x \mathrm{Ch} x;$$

and

$$\mathrm{S} \varpi \varrho'_1 \varrho'_2 = \mathrm{S} \varrho \varrho'_1 \varrho'_2 - m \mathrm{S} \alpha \varrho'_1 \varrho'_2,$$

the origin being changed to any assumed point on α; whence easily

$$V = \Phi \frac{abc}{3} \int_x (\mathrm{Sh} x - m \mathrm{Sh} x \mathrm{Ch} x)$$

$$= \Phi \frac{abc}{3} \Big[\mathrm{Ch} x - \frac{m}{2} \mathrm{Sh}^2 x \Big]_{x_0}^{x}. \qquad (57)$$

Determining m by the condition $m = \mathrm{Ch} x$, where x is the superior and 0 the lower limit, this formula becomes

$$V = \Phi \frac{abc}{3} \Big[\mathrm{Ch} x - \tfrac{1}{2} \mathrm{Ch} x \, \mathrm{Sh}^2 x - 1 \Big]; \qquad (58)$$

and gives a portion of the *interior* volume, determined by a plane section perpendicular to α.

The unparted hyperboloid can be treated in a somewhat similar manner. In this case, we may assume

$$\varpi = \varrho - \sigma \mathrm{Ch} x = \alpha \mathrm{Sh} x,$$

which is equivalent to the condition that ϖ shall be the ordinate parallel to α for any assumed point of the surface. $\mathrm{S}\varpi\varrho_1'\varrho_2'dxdy$, being always the elementary parallelopiped four of whose edges are parallel and equal to $\alpha\mathrm{Sh}x$, will therefore represent the element of the volume which we are seeking. By easy transformations, we find

$$\mathrm{D}_x\mathrm{D}_y V = \mathrm{S}\varpi\varrho_1'\varrho_2' = \mathrm{Sh}^2x\mathrm{Ch}x\mathrm{S}\alpha\sigma\sigma'$$

$$= \mathrm{Sh}^2x\mathrm{Ch}x\mathrm{S}\alpha\beta\gamma = ahc\mathrm{Sh}^2x\mathrm{Ch}x,$$

whence

$$V = abc\int_x\int_y \mathrm{Sh}^2x\mathrm{Ch}x = \Phi\,\frac{abc}{3}\Big[\mathrm{Sh}^3x\Big]_{x_0}^{x},$$

and if we take zero as the lower limit,

$$V = \Phi\,\frac{abc}{3}\Big[\mathrm{Sh}^3x\Big],$$

which gives an expression for the volume contained between the surface and the elliptical right cylinder whose axis is α and whose base is the section determined by the vector $\sigma\mathrm{Ch}x$.

Cambridge, June 1, 1877.